defining connectedness in the stratifications ρ_1 and ρ_2.

In the example under consideration, after reduction the manifold V_{12} is five-dimensional, the manifolds V_1 and V_2 are three-dimensional, and the base V is two-dimensional. The distribution Δ extended over the vector fields Y_1 and Y_2 is two-dimensional and the distributions Δ_1, Δ_2 and Δ_{12}, defined by the operators $\partial/\partial\tau_1, \partial/\partial\tau_2$ and $\partial/\partial\tau$, respectively, are one-dimensional. The three Pfaff equations presented above define the distribution Δ. The second and third of them define the distribution Δ_1, and the first and third, the distribution Δ_2.

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CONDITIONS FOR FINITENESS OF THE NUMBER OF INSTABILITY ZONES IN THE PROBLEM OF NORMAL VIBRATIONS OF NON-LINEAR SYSTEMS*

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Conservative non-linear systems with two degrees of freedom that allow of normal vibrations with rectilinear trajectories in configuration space are examined. The normal vibrations of non-linear systems are a generalization for normal (principal) vibrations of linear systems /1/. The value of such solutions is determined by the fact that the resonance modes are close to normal vibrations for small external periodic effects.

A number of recent papers (/2—5/ etc.) are devoted to the analysis of normal vibrations. Within the framework of the stability problem to a first approximation, or normal vibrations, conditions are obtained for which the number of instability zones in the system parameter space is finite. The eigenfunctions and eigenvalues corresponding to the zone boundaries are determined.

1. Let the motion of a conservative system be determined by the equations

$$x_i^{\cdot\cdot} + \partial\Pi/\partial x_i = 0 \quad (i = 1, 2) \tag{1.1}$$

where $\Pi(x_1, x_2)$ is a positive-definite potential.

We assume that the system allows normal vibrations $x_2 = Cx_1$ (C is a constant). Such systems are described in /1, 3, 5/. Rotation of the coordinate axes can always result in a solution in the form $x_2 = 0$, and a system potential in the form

$$\Pi(x_1, x_2) = \sum_{i=2}^{m} a_i x_1^i + x_2^2 \sum_{i=0}^{m-2} e_i x_1^i + \sum_{i=3}^{\infty} x_2^i g_i(x_1)$$

The condition for the existence of the solutions mentioned $\partial\Pi(x_1, 0)/\partial x_2 = 0$ is satisfied identically.

Motion in time along the normal vibrations trajectory is described by a second-order equation

$$x^{\cdot\cdot} + \partial\Pi(x, 0)/\partial x = 0, \quad x \equiv x_1 \tag{1.2}$$

where the first integral (the energy integral) has the form

$$x^{\cdot 2}/2 + \Pi(x, 0) = h \tag{1.3}$$

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The orbital stability of the rectilinear vibrations mode is related to variations in y orthogonal to the trajectory. To a first approximation, the equation describing the motion in the orthogonal direction has the form

$$y^{\cdot\cdot} + y\,(\partial^2\Pi\,(x, 0)/\partial x_2^2) = 0 \tag{1.4}$$

Conditions for the finiteness of the number of instability zones are obtained in /6/ for equations of the form (1.4). Namely, periodic potentials $u\,(t)$ of the Schrödinger equation

$$-[-d^2/dt^2 + u\,(t)]\,\psi = e\psi$$

(in the system under consideration $u - e = -\partial^2\Pi\,(x, 0)/\partial x_2^2$, $e = -2e_0$) have just n finite forbidden bands (instability zones) if the equation $i\chi^{\cdot} - \chi^2 = u - e$ admits of a solution of the form $\chi = (D - ip^{\cdot}/2)/p$, where D is a constant, and p is a polynomial in e of degree n with variable coefficients. The equation to determine the polynomial p is written in the following form

$$p^{\cdot\cdot\cdot} - 4\,(u - e)\,p^{\cdot} - 2u^{\cdot}p = 0 \tag{1.5}$$

It is best to use the well-known law of motion in the normal form $x\,(t)$ to solve the problem, and to obtain an equation with regular singularities instead of an equation with periodic coefficients. In place of t we introduce a new independent x ("algebraization in the Ince sense" /7/) by using relationships (1.2), (1.3) and the equation $x^{\cdot\cdot} = -(\partial^2\Pi\,(x, 0)/\partial x^2)\,x^{\cdot}$, which follows from (1.2).

In place of (1.5) we now obtain the equation

$$2p'''\,(h - \Pi\,(x, 0)) + 3p''\left(-\frac{\partial\Pi\,(x, 0)}{\partial x}\right) + p'\left(-\frac{\partial^2\Pi\,(x, 0)}{\partial x^2} + 4\frac{\partial^2\Pi\,(x, 0)}{\partial x_2^2}\right) + 2p\frac{\partial}{\partial x}\frac{\partial^2\Pi\,(x, 0)}{\partial x_2^2} = 0 \tag{1.6}$$

where the prime denotes differentiation with respect to x. Substituting

$$p = \sum_{k=0}^{n} \alpha_k\,(x)\,e^k \tag{1.7}$$

into (1.6) and grouping terms containing identical powers of e, we obtain a problem in the eigenvalues a_i, e_i. Since the potential $\Pi\,(x, 0)$ and the derivatives of the potential in (1.6) are polynomials in x, the function $\alpha_k\,(x)$ must also be sought in the form of a polynomial x.

Omitting the awkward intermediate calculations, we present just the final results for certain classes of systems.

For a system "linear with cubes" ($a_2 \neq 0$, $a_4 \neq 0$, $e \neq 0$, $e_2 \neq 0$, all the remaining a_i, e_i coefficients vanish) the condition for the existence of the solution (1.7) has the form

$$e_2 = n\,(n + 1)\,a_4 \tag{1.8}$$

This result is evident since the variational equation in this case is the Lamé equation.

For a system "linear with cubes and a fifth degree" ($a_2 \neq 0$, $a_4 \neq 0$, $a_6 \neq 0$, $e \neq 0$, $e_2 \neq 0$, $e_4 \neq 0$, the remaining coefficients vanish), the appropriate conditions have the form

$$e_2 = 2n\,(n + 1)\,a_4,\ \ e_4 = 4n\,(n + 1)\,a_6,\ \ 4a_2a_4a_6 - a_4^3 + 8ha_6^2 = 0. \tag{1.9}$$

For a system "linear with cubes and squares" ($a_2 \neq 0$, $a_3 \neq 0$, $a_4 \neq 0$, $e \neq 0$, $e_1 \neq 0$, $e_2 \neq 0$, the remaining a_i, e_i coefficients vanish), we obtain

$$e_2 = n\,(n + 1)\,a_4,\quad e_1 = n\,(n + 1)\,a_3/2,\quad a_3^2 = 4a_2a_4 \tag{1.10}$$

2. To clarify the sense of the relationships (1.8)–(1.10), we apply Ince algebraization directly to the variational equation (1.4). We consequently obtain a generalized Lamé equation, namely an equation of the Fuchs class whose exponents of the singularities equal 0 and ½. For a definite symmetry in the arrangement of the singularities and a symmetry of the auxiliary parameters, this equation is reduced to a generalized Lamé equation with a lower number of singularities by a quadratic transformation.

The simplest equation that can be obtained in this manner is the standard Lamé equation in algebraic form with real coefficients

$$y''\,(z^2 - a^2)\,(z^2 - b^2) + y'z\,(2z^2 - a^2 - b^2) - y\,(n\,(n + 1)\,z^2 - \lambda) = 0 \tag{2.1}$$

The coefficient of y'' has the meaning of the kinetic energy of the transformed system, which vanishes twice per period at the amplitude values $z = \pm a$ for the periodic regime under consideration. Consequently, without loss of generality, we assume that $a > 0, |z| \leqslant a$ and either $b^2 < 0$, or $b^2 > a^2$.

Let us write the eigenfunctions and eigenvalues of problems determining the stability boundaries to a first approximation in the case of n isntability zones in the order of increasing growth of the number of zeros of the eigenfunctions $n = 0, 1, 2$.

The following intermediate notation is introduced here

$$\alpha_\pm = 2\,\{a^2 + b^2 \pm ((a^2 + b^2)^2 - 3a^2b^2)^{1/2}\},\quad \xi = \sqrt{a^2 - z^2}$$
$$\eta_1 = \sqrt{z^2 - b^2},\quad \eta_2 = \sqrt{b^2 - z^2},\quad \zeta_i = z^2 - 2a^2b^2/\lambda_i\ \ (i = 0, 4)$$

The eigenfunctions and eigenvalues have the following form (C is an arbitrary constant)

$$
\begin{aligned}
&n = 0: y_0 = C, \quad \lambda_0 = 0\\
&n = 1: \ (b^2 < 0)\ y_0 = C\eta_1, \quad \lambda_0 = a^2\\
&y_1 = Cz,\ \lambda_1 = a^2 + b^2; \quad y_2 = C\xi, \quad \lambda_2 = b^2\\
&n = 1: \ (b^2 > a^2)\ y_0 = C\eta_2, \quad \lambda_0 = a^2\\
&y_1 = C\xi,\ \lambda_1 = b^2; \quad y_2 = Cz, \quad \lambda_2 = a^2 + b^2\\
&n = 2: \ (b^2 < 0)\ y_0 = C\zeta_0, \quad \lambda_0 = \alpha_+\\
&y_1 = Cz\eta_1, \quad \lambda_1 = 4a^2 + b^2; \quad y_2 = C\xi\eta_1, \quad \lambda_2 = a^2 + b^2\\
&y_3 = Cz\xi, \quad \lambda_3 = a^2 + 4b^2; \quad y_4 = C\zeta_4, \quad \lambda_4 = \alpha_-\\
&n = 2: \ (b^2 > a^2)\ y_0 = C\zeta_0, \quad \lambda_0 = \alpha_-\\
&y_1 = C\xi\eta_2, \quad \lambda_1 = a^2 + b^2; \quad y_2 = Cz\eta_2,\ \lambda_2 = 4a^2 + b^2\\
&y_3 = Cz\xi, \quad \lambda_3 = a^2 + 4b^2; \quad y_4 = C\zeta_4, \quad \lambda_4 = \alpha_+
\end{aligned}
$$

We note that the intervals

$$(-\infty, \lambda_0],\ [\lambda_1, \lambda_2],\ [\lambda_3, \lambda_4]$$

correspond to the instability domains.

Equation (2.1) corresponds to (1.4) for the system "linear with cubes" (in the case of finite zones) if

$$
\begin{aligned}
&h/a_4 = a^2b^2; \quad e_4/a_4 = -\lambda\\
&a_2/a_4 = -a^2 - b^2; \quad e_2/a_4 = n(n+1); \quad x = z
\end{aligned}
$$

A shift along the real axis $z = z_1 + \mu$, where $\mu = ((a^2 + b^2)/2)^{1/2}$ results in a variational equation for the system "linear with cubes and squares" (the case of a finite number of instability zones) for

$$
\begin{aligned}
&h/a_4 = -a^2b^2 + \mu^2(a^2 + b^2) - \mu^4; \quad e_0/a_4 = -\lambda + \mu^2 n(n+1)\\
&a_2/a_4 = 2(a^2 + b^2); \quad e_1/a_4 = 2\mu n(n+1)\\
&a_3/a_4 = 4\mu; \quad e_2/a_4 = n(n+1); \quad x = z_1
\end{aligned}
$$

Equalities (1.10) hence follow.

By using the quadratic transformation $z = z_2^2 - a$ we obtain a variational equation for the system "linear with cubes and a fifth power" (the case of a finite number of instability zones) for

$$
\begin{aligned}
&h/a_6 = 2a(a^2 - b^2); \quad e_0/a_6 = 4(a^2 n(n+1) - \lambda)\\
&a_2/a_6 = 5a^2 - b^2; \quad e_2/a_6 = -8an(n+1)\\
&a_4/a_6 = -4a; \quad e_4/a_6 = 4n(n+1); \quad x = z_2
\end{aligned}
$$

Equalities (1.9) hence follow.

The eignefunctions and eigenvalues are transformed correspondingly.

It is interesting that in the last case, unlike those preceding, only the instability zones bounded by the eigenfunctions having an even number of zeros are conserved. The remaining instability zones shrink to a line. In particular, the first, ordinarily the widest, instability domain (as for instance in the Mathieu equation) shrinks into a line.

If the equalities (1.8)–(1.10) are only satisfied approximately, then there is an infinite number of instability zones; however, except for those extracted above, they are all "narrow" in a definite sense since they shrink to a line when conditions (1.8)–(1.10) are satisfied.

An an illustration, a plane entirely elastic vibrational loop with two degrees of freedom /8/ is considered. Problems of rod dynamics, guy-rope structures with lumped elements, etc. can result in such kinds of models. We consider the plane transverse vibrations of two single point masses interconnected by a linear spring with stiffness c_2 and length L in the equilibrium state, and connected to fastening points by linear springs with stiffness c_1 and lengths l in the equilibrium state. It is assumed that the springs are prestrained in such a manner that constant forces T (tensile or compressive) are applied at the fastening points. The Hamiltonian of the system is

$$
\begin{aligned}
&H = \frac{1}{2}\left\{\sum_{i=1}^{2} x_i^{\cdot 2} + \sum_{i=1}^{2} c_1((x_i^2 + l^2)^{1/2} - l_0)^2 + c_2(((x_1 - x_2)^2 + L^2)^{1/2} - L_0)^2\right\}\\
&l_0 = l - T/c_1,\ L_0 = L - T/c_2
\end{aligned}
$$

Here x_i are the transverse displacements of the mass.

Retaining terms containing just the first, third, and fifth degrees in x_1, x_2 in the equations of motion, we arrive at the following equations

$$
\begin{aligned}
&x_i^{\cdot\cdot} + c_1\left((1 - l_0/l)x_i + \frac{l_0}{2}\left(\frac{x_i}{l}\right)^3 - \frac{3l_0}{8}\left(\frac{x_i}{l}\right)^5\right) +\\
&c_2\left(\left(1 - \frac{L_0}{L}\right)(x_i - x_j) + \frac{L_0}{2}\left(\frac{x_i - x_j}{L}\right)^3 - \frac{3L_0}{8}\left(\frac{x_i - x_j}{L}\right)^5\right) = 0 \quad (i = 1, 2;\ j = 3 - i)
\end{aligned}
$$

We use the notation

$$T/l = \tau_1,\ \ T/L = \tau_2,\ \ c_1 l_0/l = \gamma_1,\ \ c_2 L_0/L = \gamma_2,\ \ (l/L)^2 = \sigma$$

and consider the antiphase vibrations mode $x_1 = -x_2$. Changing to the new variables

$x = (x_1 - x_2)/(2l),\ y = (x_1 + x_2)/(2l)$, we obtain an equation describing the motion in the mode $y = 0, x = x(t)$ (analogous to (1.2)) and an equation governing the orbital stability to a first approximation (analogous to (1.4)), which have the following form

$$x^{\cdot\cdot} + 2a_2x + 4a_4x^3 + 6a_6x^5 = 0$$
$$y^{\cdot\cdot} + 2(e_0 + e_2x^2 + e_4x^4)\,y = 0$$
$$a_2 = \tau_1/2 + \tau_2,\quad a_4 = \gamma_1/8 + \gamma_2\sigma,\quad a_6 = -\gamma_1/16 - 2\gamma_2\sigma^2$$
$$e_0 = \tau_1/2,\quad e_2 = 3\gamma_1/4,\quad e_4 = -15\gamma_1/16$$

In this case conditions (1.9) result in the relationships

$$3 = N(1 + 8\gamma_3),\quad 15 = 4N(1 + 32\gamma_3\sigma)$$
$$8(\tau_1 + 2\tau_2)(1 + 8\gamma_3)(1 + 32\gamma_3\sigma) + (1 + 8\gamma_3)^3 = 4h(1 + 32\gamma_3\sigma)^3$$
$$\gamma_3 = \gamma_2\sigma/\gamma_1,\quad N = n(n + 1)$$

Here n is the number of bounded instability zones $n = 0, 1, 2, \ldots$

For instance, let $n = 1$. Then there should be $\sigma = 7/16, \gamma_3 = 1/16, 40(\tau_1 + 2\tau_2) + 6 = 25h$.

Near the values of the system parameters and the energy h that satisfy the above relationships, all except n instability zones are "narrow" since they shrink to a line in the case when the relationships are satisfied exactly. This also refers to the first parametric resonance zone, which is ordinarily assumed to be "wide".

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ON THE EXTREMALITY HYPOTHESIS OF STABLE RESONANCE MOTIONS*

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On the basis of a proposed approximate method of determining the mean values of functions of the coordinates and time on almost-integrable trajectories of dynamic systems, the force function and kinetic energy are averaged in the following problems: the motion of a material point in the neighbourhood of triangular points of libration of the plane circular restricted three-body problem, the motion of a physical pendulum with a rapidly oscillating point of suspension in the neighbourhoods of the lower and upper equilibrium positions. Preference is shown for the following hypotheses: the minimum of the averaged potential (V.V. Beletskii hypothesis), kinetic, and total energy of the mechanical system at stable, isolated, synchronous motions.

The extremal principle proposed in the form of the V.V. Beletskii hypothesis /2, 3/ is of special interest in investigations of the extremal properties of stable resonance (synchronous) motions: The function

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